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CONTACT BETWEEN A MOVING STAMP AND AN ELASTIC HALF-PLANE WHEN THERE IS WEAR*

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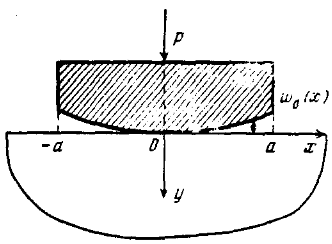
The plane contact problem of elasticity theory is considered for a half-plane when there is abrasive wear. It is assumed in the problems considered by Galin /1/ that the stamp moves in certain guides ensuring there is no closure between the contacting bodies. A solution is given below for one of the problems formulated by Galin, without taking account of the simplifying assumptions mentioned.

1. The plane problem of elasticity theory of the pressure of a stamp with profile $w_0 = Ax^2$ (A is a given constant) on a half-plane is considered (see the sketch). The stamp undergoes a displacement in a direction parallel to the contact strip whereupon wear of the half-plane occurs. We assume that there is no friction between the stamp and the half-plane in a direction perpendicular to the stamp motion (in the direction of the x-axis). It is required to determine the time-varying contact pressure and the settling of the stamp.



Let $w_1(x, t)$ be the displacement of half-plane boundary points along the y-axis due to wear, $w_2(x, t)$ the elastic displacement of the half-plane boundary points, and $\delta(t)$ the settling of the stamp. We have from the contact condition between the stamp and the half-plane in the section $[-a, a]$

$$Ax^2 + w_1(x, t) + w_2(x, t) = \delta(t) \quad (-a < x < a,\ t \geqslant 0). \tag{1.1}$$

Taking into account according to /1/ that

$$w_1(x, t) = k\int_0^t p(x, \tau)\, d\tau \tag{1.2}$$

$$w_2(x, t) = B\int_{-a}^{a} \ln\frac{1}{|x-\xi|}\, p(\xi, t)\, d\xi, \quad B = \frac{2(1-\nu^2)}{\pi E}$$

where $p(x, t)$ is the desired contact pressure, k is a coefficient characterizing the half-plane wear, E is Young's modulus, and ν is Poisson's ratio of the half-plane material, we obtain from (1.1)

$$B\int_{-a}^{a} \ln\frac{1}{|x-\xi|}\, p(\xi, t)\, d\xi + k\int_0^t p(x, \tau)\, d\tau = \delta(t) - Ax^2 \tag{1.3}$$
$$(-a < x < a,\ t \geqslant 0).$$

In addition to (1.3), the contact pressure $p(x, t)$ should satisfy the equilibrium condition

$$\int_{-a}^{a} p(x, t)\, dx = P \quad (t \geqslant 0). \tag{1.4}$$

We assume thatt the contact area $[-a, a]$ does not vary with time, which is known to hold for a sufficiently large force P assuring total insertion of the stamp into the half-plane at $t = 0$. For this the inequality /1/

$$s = a^2A/(BP) \leqslant 1 \tag{1.5}$$

should be satisfied.

As is well-known /1/, under the condition (1.5)

$$p(x, 0) = P\,\frac{1 + s - 2s(x/a)^2}{\pi\sqrt{a^2 - x^2}} \quad (-a < x < a). \tag{1.6}$$

Thus, it is required to solve (1.3) for $p(x, t)$ under the additional conditions (1.4) and (1.6) by taking into account that the settling of the stamp $\delta(t)$ is also unknown.

2. Setting $t = 0$ in (1.3) and subtracting the equation obtained from (1.3), we will have

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$$B \int_{-a}^{a} \ln \frac{1}{|x-\xi|} [p(\xi,t) - p(\xi,0)]\,d\xi + \tag{2.1}$$
$$k \int_{0}^{t} p(x,\tau)\,d\tau = \delta(t) - \delta(0) \quad (-a < x < a, t \geqslant 0).$$

To eliminate the unknown right side from (2.1), we integrate (2.1) with respect to x in the interval $[-a, a]$. Changing the order of integration in the double integrals obtained and taking account of the equilibrium condition (1.4), we find

$$\delta(t) - \delta(0) = \frac{B}{2a} \int_{-a}^{a} dx \int_{-a}^{a} \ln \frac{1}{|x-\xi|} [p(\xi,t) - p(\xi,0)]\,d\xi + \frac{kPt}{2a}. \tag{2.2}$$

Eliminating the difference $\delta(t) - \delta(0)$ from (2.1) by using the relationship (2.2), we arrive at the equation

$$B \int_{-a}^{a} \left[\ln \frac{1}{|x-\xi|} - \frac{1}{2a} \int_{-a}^{a} \ln \frac{1}{|x-\xi|}\,dx\right] [p(\xi,t) - p(\xi,0)]\,d\xi + \tag{2.3}$$
$$k \int_{0}^{t} p(x,\tau)\,d\tau = \frac{kPt}{2a}.$$

We now make the following natural assumption from the viewpoint of mechanics: the contact pressure is equalized during wear, i.e., we seek it in the form

$$p(x,t) = P/(2a) + q(x,t) \tag{2.4}$$

$$q(x,t) \underset{(t\to\infty)}{\longrightarrow} 0 \; (-a < x < a), \qquad \int_{-a}^{a} q(x,t)\,dx = 0 \quad (t \geqslant 0). \tag{2.5}$$

The last condition results from (1.4). Substituting (2.4) into (2.3), we obtain an equation to determine the new unknown function $q(x,t)$

$$B \int_{-a}^{a} \left[\ln \frac{1}{|x-\xi|} - \frac{1}{2a} \int_{-a}^{a} \ln \frac{1}{|x-\xi|}\,dx\right] [q(\xi,t) - q(\xi,0)]\,d\xi + \tag{2.6}$$
$$k \int_{0}^{t} q(x,\tau)\,d\tau = 0 \quad (-a < x < a, t \geqslant 0).$$

We solve the integral equation (2.6) by separation of variables. Taking the first condition in (2.5) into account, we seek the particular solution of (2.6) in the form

$$q(x,t) = \varphi(x) \exp(-k\lambda t/B), \quad \lambda > 0; \quad \int_{-a}^{a} \varphi(x)\,dx = 0 \tag{2.7}$$

(the integral condition follows from the last condition in (2.5)). Substituting expression (2.7) into (2.6), we obtain

$$\varphi(x) - \lambda \int_{-a}^{a} \left[\ln \frac{1}{|x-\xi|} - \frac{1}{2a} \int_{-a}^{a} \ln \frac{1}{|x-\xi|}\,dx\right] \varphi(\xi)\,d\xi = 0 \tag{2.8}$$
$$(-a < x < a).$$

We note that when the function $\varphi(x)$ satisfies this integral equation the integral condition (2.7) is automatically satisfied; this can be seen by integrating both sides of (2.8) with respect to x in the interval $[-a, a]$.

After seeking the eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ $(n = 1, 2, \ldots)$ of (2.8), we represent the desired function $q(x,t)$ by the series

$$q(x,t) = \sum_{n=1}^{\infty} c_n \varphi_n(x) \exp\left(-\frac{k\lambda_n}{B} t\right) \quad (-a < x < a, t \geqslant 0). \tag{2.9}$$

The coefficients c_n of series (2.9) are found, as follows from (2.4) and (1.6), from the expansion

$$q(x,0) = P\left[\frac{1-\varepsilon-2\varepsilon(x/a)^2}{\pi\sqrt{a^2-x^2}} - \frac{1}{2a}\right] = \sum_{n=1}^{\infty} c_n \varphi_n(x) \; (-a < x < a). \tag{2.10}$$

3. We henceforth take the quantity a as the unit of length, i.e., we consider $a = 1$ in all the preceding and subsequent formulas. For $a = 1$ we shall seek the eigenvalues and eigenfunctions of (2.8) by Galerkin's method

$$\varphi(x) \approx \frac{1}{\sqrt{1-x^2}} \sum_{i=0}^{N} a_i T_{2i}(x) \quad (-1 < x < 1) \tag{3.1}$$

where $T_i(x)$ are Chebyshev polynomials of the first kind. Selection of the form (3.1) for the function $\varphi(x)$ is advisable because of the presence of the spectral ralation

$$\int_{-1}^{1} \ln \frac{1}{|x-\xi|} \frac{T_i(\xi)}{\sqrt{1-\xi^2}} d\xi = \sigma_i T_i(x) \quad (-1 < x < 1) \tag{3.2}$$
$$\sigma_0 = \pi \ln 2, \quad \sigma_i = \pi/i \quad (i = 1, 2, \ldots)$$

as well as because of the known fact of the orthogonality of Chebyshev polynomials of the first kind

$$\int_{-1}^{1} \frac{T_i(x) T_j(x)}{\sqrt{1-x^2}} dx = \varepsilon_i \delta_{ij} \quad (i, j = 0, 1, \ldots) \tag{3.3}$$
$$\varepsilon_0 = \pi, \quad \varepsilon_i = \pi/2 \quad (i = 1, 2, \ldots), \quad \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Only the even polynomials $T_{2i}(x)$ are in the expansion (3.1) since the function $\varphi(x)$ must be even.

We substitute (3.1) into (2.8) for $a = 1$ and use the spectral relationship (3.2). Then multiplying both sides of the equation obtained by $T_{2j}(x)(j = 0, \ldots, N)$ and integrating with respect to x between -1 and 1, by taking account of the orthogonality condition (3.3) we arrive at the homogeneous algebraic system

$$a_j - \lambda \sum_{i=0}^{N} b_{ji} a_i = 0 \quad (j = 0, 1, \ldots N). \tag{3.4}$$

Here

$$b_{ji} = \frac{\sigma_{2i}}{\varepsilon_{2j}} \left[\int_{-1}^{1} T_{2i}(x) T_{2j}(x) dx - \frac{1}{2} \int_{-1}^{1} T_{2i}(x) dx \int_{-1}^{1} T_{2j}(x) dx \right] \tag{3.5}$$
$$(i, \ j = 0, \ 1, \ldots, N).$$

Because of the identity $T_m(\cos\varphi) = \cos m\varphi \ (m = 0, 1, \ldots)$, the integrals in (3.5) are evaluated explicitly. Hence, for the coefficients $b_{ji} \ (i, j = 0, 1, \ldots, N)$ we arrive at the expression

$$b_{ji} = \frac{\sigma_{2i}}{\varepsilon_{2j}} d_{ji}, \quad d_{ji} = \gamma_{i-j} + \gamma_{i+j} - 2\gamma_i \gamma_j, \quad \gamma_m = -\frac{1}{4m^2 - 1} \tag{3.6}$$
$$(m = 0, \pm 1, \ldots).$$

Since $b_{0i} \equiv 0$ according to (3.6), it follows from system (3.4) that $a_0 = 0$. The expansion (3.1) then takes the form

$$\varphi(x) \approx \frac{1}{\sqrt{1-x^2}} \sum_{i=1}^{N} a_i T_{2i}(x) \quad (-1 < x < 1). \tag{3.7}$$

The coefficients a_i in this expansion are found from the solution of the homogeneous algebraic system (3.4) which will be converted to the following form by the introduction of the new unknowns $x_i = a_i/\sqrt{i} \ (i = 1, 2, \ldots, N)$

$$x_j - \lambda \sum_{i=1}^{N} \frac{d_{ji}}{\sqrt{ji}} x_i = 0 \quad (j = 1, 2, \ldots N) \tag{3.8}$$

with a matrix already symmetric. As is well-known, all N eigenvalues $\lambda_n \ (n = 1, 2, \ldots, N)$ of such a system are real with corresponding N linearly independent column eigenvectors

$$x_{ni} = (x_{n1}, x_{n2}, \ldots, x_{nN}) \ (n = 1, 2, \ldots, N). \tag{3.9}$$

The eigenvalues $\lambda_n \ (n = 1, 2, \ldots, N)$ of system (3.8) are also approximately the desired eigenvalues of the integral equation (2.8) for $a = 1$ (the greater N the more exact the eigenvalues), and the eigenfunctions $\varphi_n(x) \ (n = 1, 2, \ldots, N)$ are found from the formula

$$\varphi_n(x) \approx \frac{1}{\sqrt{1-x^2}} \sum_{i=1}^{N} a_{ni} T_{2i}(x) \quad (-1 < x < 1) \tag{3.10}$$

following from (3.7), where $a_{ni} = \sqrt{i}\, x_{ni}$.

Having found λ_n and $\varphi_n(x)$ $(n = 1, 2, \ldots, N)$, by using (2.9) we can also write an approximate expression for the function $q(x, t)$

$$q(x,t) \approx \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^{N} \sum_{i=1}^{N} c_n a_{ni} T_{2i}(x) \exp\left(-\frac{k\lambda_n}{B} t\right) \tag{3.11}$$
$$(-1 < x < 1,\ t \geqslant 0).$$

The unknown coefficients c_n $(n = 1, 2, \ldots, N)$ are found from the expansion (2.10) which yields for $a = 1$

$$\frac{1}{\sqrt{1-x^2}} \sum_{i=1}^{N} \beta_i T_{2i}(x) \approx \frac{P}{\pi\sqrt{1-x^2}} \left[1 + s - 2sx^2 - \frac{\pi}{2}\sqrt{1-x^2}\right]. \tag{3.12}$$

Here

$$\beta_i = \sum_{n=1}^{N} a_{ni} c_n \quad (i = 1, 2, \ldots, N). \tag{3.13}$$

We note that the approximate equality (3.12), obtained by comparing the approximate and exact expressions for $p(x, 0)$ goes over into the exact value for all $-1 < x < 1$ for $N = \infty$. Indeed, for $N = \infty$ it is transformed into a Fourier trigonometric series expansion of given continuous and continuously differentiable functions after the substitutions $x = \cos\varphi$, $T_{2i}(\cos\varphi) = \cos 2i\varphi$.

The coefficients β_i should be found from the expansion (3.12), after which we find the coefficients c_n $(n = 1, 2, \ldots, N)$ of (3.11) by solving the inhomogeneous system (3.13). The matrix of system (3.13) is non-singular since it is formed from N linearly independence column-vectors $a_{ni} = \sqrt{i}\, x_{ni}$, and therefore, the system has a unique solution.

To find the quantities β_i we multiply both sides of (3.12) by $T_{2j}(x)$ $(j = 1, 2, \ldots, N)$ and we integrate the result with respect to x between -1 and 1. Taking into account that $T_0(x) = 1$, $T_2(x) = 2x^2 - 1$ so that $1 - s - 2sx^2 = T_0(x) - sT_2(x)$ and using the orthogonality property (3.3), we find

$$\beta_1 = \frac{P}{\pi}\left(\frac{2}{3} - s\right), \quad \beta_i = \frac{2P}{\pi(4i^2-1)} \quad (i = 2, 3, \ldots, N). \tag{3.14}$$

Having found the function $q(x, t)$, the desired contact pressure $p(x, t)$ is also found using (2.4), and in addition, the change in stamp settling with time using (2.2). From (3.11), we find

$$\delta(t) - \delta(0) \approx \frac{kPt}{2} + \frac{B\pi}{2} \sum_{n=1}^{N} \sum_{i=1}^{N} \frac{c_n a_{ni}}{i(4i^2-1)} \left[1 - \exp\left(-\frac{k\lambda_n}{B} t\right)\right]. \tag{3.15}$$

Confining ourselves in particular to the simplest case $N = 1$, we obtain to a first approximation

$$\lambda_1 = \frac{45}{32}; \quad p(x,t) = \frac{P}{\pi\sqrt{1-x^2}} \times \tag{3.16}$$
$$\left[\frac{\pi}{2}\sqrt{1-x^2} - \left(\frac{2}{3} - s\right)(2x^2 - 1)\exp\left(\frac{k\lambda_1}{B} t\right)\right]$$
$$\delta(t) - \delta(0) = PB\left[\frac{kt}{2B} + \frac{1}{6}\left(\frac{2}{3} - s\right)\left(1 - \exp\left(\frac{k\lambda_1}{B} t\right)\right)\right].$$

Comparing the approximation expression (3.16) and the exact expression (1.6) for $p(x, 0)$ we see that the pressure at the centre of the contact area, say, given by (3.16) differs from the exact value by not more than 10% for all $0 < s \leqslant 1$. This error increases somewhat as one approaches the edge of the contact area. To reduce it, the first approximation formula (3.16) should be replaced by a more exact formula by increasing the value of N.

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